

# Lectures on the Epsilon Calculus

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## Introduction

These notes were written for use in a short course on the epsilon calculus I taught at the Vienna University of Technology as an Erasmus Mundus Fellow in 2009. I had at one point hoped to expand them substantially, and this may yet happen. In the meantime, perhaps they prove useful to someone even if the results presented remain quite basic, proofs are left as exercises, explanations are sparse (to say the least), and there are no references (but see [here](#)). Semantics for the epsilon calculus, completeness proofs, as well as proofs of the first epsilon theorem, are, after all, still hard to find, at least in English. I claim no credit to the results contained herein; in particular, I've learned the intensional and extensional semantics and completeness results from Grigori Mints' lectures on the epsilon calculus. Please [contact me](#) if you find a mistake.

**Exercise 1.** Find the mistakes in these notes.

# 1 Syntax

## 1.1 Languages

**Definition 1.1.** The language of the elementary calculus  $L_{EC}$  contains the following symbols:

1. Variables Var:  $x_0, x_1, \dots$
2. Function symbols  $Fct^n$  of arity  $n$ , for each  $n \geq 0$ :  $f_0^n, f_1^n, \dots$
3. Predicate symbols  $Pred^n$  of arity  $n$ , for each  $n \geq 0$ :  $P_0^n, P_1^n, \dots$
4. Identity:  $=$
5. Propositional constants:  $\perp, \top$ .
6. Propositional operators:  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ .
7. Punctuation: parentheses:  $(, )$ ; comma:  $,$

For any language  $L$ , we denote by  $L^-$  the language  $L$  without the identity symbol, by  $L_\varepsilon$  the language  $L$  plus the symbol  $\varepsilon$ , and by  $L_\forall$  the language  $L$  plus the quantifiers  $\forall$  and  $\exists$ . We will usually leave out the subscript EC, and write  $L_\forall$  for the language of the predicate calculus,  $L_\varepsilon$  for the language of the  $\varepsilon$ -calculus, and  $L_{\varepsilon\forall}$  for the language of the extended epsilon calculus.

**Definition 1.2.** The *terms* Trm and *formulas* Frm of  $L_{\varepsilon\forall}$  are defined as follows.

1. Every variable  $x$  is a term, and  $x$  is free in it.
2. If  $t_1, \dots, t_n$  are terms, then  $f_i^n(t_1, \dots, t_n)$  is a term, and  $x$  occurs free in it wherever it occurs free in  $t_1, \dots, t_n$ .
3. If  $t_1, \dots, t_n$  are terms, then  $P_i^n(t_1, \dots, t_n)$  is an (atomic) formula, and  $x$  occurs free in it wherever it occurs free in  $t_1, \dots, t_n$ .
4.  $\perp$  and  $\top$  are formulas.
5. If  $A$  is a formula, then  $\neg A$  is a formula, with the same free occurrences of variables as  $A$ .
6. If  $A$  and  $B$  are formulas, then  $(A \wedge B)$ ,  $(A \vee B)$ ,  $(A \rightarrow B)$ ,  $(A \leftrightarrow B)$  are formulas, with the same free occurrences of variables as  $A$  and  $B$ .
7. If  $A$  is a formula in which  $x$  has a free occurrence but no bound occurrence, then  $\forall x A$  and  $\exists x A$  are formulas, and all occurrences of  $x$  in them are bound.
8. If  $A$  is a formula in which  $x$  has a free occurrence but no bound occurrence, then  $\varepsilon_x A$  is a term, and all occurrences of  $x$  in it are bound.

The terms  $\text{Trm}(L)$  and formulas  $\text{Frm}(L)$  of a language  $L$  are those terms and formulas of  $L_{\varepsilon\forall}$  in the vocabulary of  $L$ .

If  $E$  is an expression (term or formula), then  $\text{FV}(E)$  is the set of variables which have free occurrences in  $E$ .  $E$  is called *closed* if  $\text{FV}(E) = \emptyset$ . A closed formula is also called a *sentence*.

When  $E, E'$  are expressions (terms or formulas), we write  $E \equiv E'$  iff  $E$  and  $E'$  are syntactically identical up to a renaming of bound variables. We say that a term  $t$  is *free for  $x$  in  $E$*  iff  $x$  does not occur free in the scope of an  $\varepsilon$ -operator  $\varepsilon_y$  or quantifier  $\forall y, \exists y$  for any  $y \in \text{FV}(t)$ .

If  $E$  is an expression and  $t$  is a term, we write  $E[x/t]$  for the result of substituting every free occurrence of  $x$  in  $E$  by  $t$ , provided  $t$  is free for  $x$  in  $E$ , and renaming bound variables in  $t$  if necessary.

If  $t$  is not free for  $x$  in  $E$ ,  $E[x/t]$  is any formula  $E'[x/t]$  where  $E' \equiv E$  and  $t$  is free for  $x$  in  $E'$ . If  $E' \equiv E[x_1/t_1] \dots [x_n/t_n]$ ,  $E'$  is called an *instance* of  $E$ .

We write  $E(x)$  to indicate that  $x \in \text{FV}(E)$ , and  $E(t)$  for  $E[x/t]$ . It will be apparent from the context which variable  $x$  is substituted for.

**Definition 1.3.** A term  $t$  is a *subterm* of an expression (term or formula)  $E$ , if for some  $E'(x)$ ,  $E \equiv E'(x)[x/t]$ . It is a *proper subterm* of a term  $u$  if it is a subterm of  $u$  but  $t \neq u$ .

A term  $t$  is an *immediate subterm* of an expression  $E$  if  $t$  is a subterm of  $E$ , but not a subterm of a proper subterm of  $E$ .

**Definition 1.4.** If  $t$  is a subterm of  $E$ , i.e., for some  $E'$  we have  $E \equiv E'[x/t]$ , then  $E\{t/u\}$  is  $E'[x/u]$ .

We intend  $E\{t/u\}$  to be the result of replacing every occurrence of  $t$  in  $E$  by  $u$ . But, the “brute-force” replacement of every occurrence of  $t$  in  $u$  may not be what we have in mind here. (a) We want to replace not just every occurrence of  $t$  by  $u$ , but every occurrence of a term  $t' \equiv t$ . (b)  $t$  may have an occurrence in  $E$  where a variable in  $t$  is bound by a quantifier or  $\varepsilon$  outside  $t$ , and such occurrences shouldn't be replaced (they are not subterm occurrences). (c) When replacing  $t$  by  $u$ , bound variables in  $u$  might have to be renamed to avoid conflicts with the bound variables in  $E'$  and bound variables in  $E'$  might have to be renamed to avoid free variables in  $u$  being bound.

**Definition 1.5** ( $\varepsilon$ -Translation). If  $E$  is an expression, define  $E^\varepsilon$  by:

1.  $E^\varepsilon = E$  if  $E$  is a variable, a constant symbol, or  $\perp$ .
2. If  $E = f_i^n(t_1, \dots, t_n)$ ,  $E^\varepsilon = f_i^n(t_1^\varepsilon, \dots, t_n^\varepsilon)$ .
3. If  $E = P_i^n(t_1, \dots, t_n)$ ,  $E^\varepsilon = P_i^n(t_1^\varepsilon, \dots, t_n^\varepsilon)$ .
4. If  $E = \neg A$ , then  $E^\varepsilon = \neg A^\varepsilon$ .
5. If  $E = (A \wedge B)$ ,  $(A \vee B)$ ,  $(A \rightarrow B)$ , or  $(A \leftrightarrow B)$ , then  $E^\varepsilon = (A^\varepsilon \wedge B^\varepsilon)$ ,  $(A^\varepsilon \vee B^\varepsilon)$ ,  $(A^\varepsilon \rightarrow B^\varepsilon)$ , or  $(A^\varepsilon \leftrightarrow B^\varepsilon)$ , respectively.

6. If  $E = \exists x A(x)$  or  $\forall x A(x)$ , then  $E^\varepsilon = A^\varepsilon(\varepsilon_x A(x)^\varepsilon)$  or  $A^\varepsilon(\varepsilon_x \neg A(x)^\varepsilon)$ .
7. If  $E = \varepsilon_x A(x)$ , then  $E^\varepsilon = \varepsilon_x A(x)^\varepsilon$ .

## 1.2 $\varepsilon$ -Types, Degree, and Rank

**Definition 1.6.** An  $\varepsilon$ -term  $p \equiv \varepsilon_x B(x; x_1, \dots, x_n)$  is a *type of an  $\varepsilon$ -term  $\varepsilon_x A(x)$*  iff

1.  $p \equiv \varepsilon_x A(x)[x_1/t_1] \dots [x_n/t_n]$  for some terms  $t_1, \dots, t_n$ .
2.  $\text{FV}(p) = \{x_1, \dots, x_n\}$ .
3.  $x_1, \dots, x_n$  are all immediate subterms of  $p$ .
4. Each  $x_i$  has exactly one occurrence in  $p$ .
5. The occurrence of  $x_i$  is left of the occurrence of  $x_j$  in  $p$  if  $i < j$ .

We denote the set of types of a language as  $\text{Typ}$ .

**Proposition 1.7.** *The type of an epsilon term  $\varepsilon_x A(x)$  is unique up to renaming of bound, and disjoint renaming of free variables, i.e., if  $p = \varepsilon_x B(x; x_1, \dots, x_n)$ ,  $p' = \varepsilon_y B'(y; y_1, \dots, y_m)$  are types of  $\varepsilon_x A(x)$ , then  $n = m$  and  $p' \equiv p[x_1/y_1] \dots [x_n/y_n]$*

*Proof.* Exercise. □

**Definition 1.8.** An  $\varepsilon$ -term  $e$  is *nested in* an  $\varepsilon$ -term  $e'$  if  $e$  is a proper subterm of  $e'$ .

**Definition 1.9.** The *degree*  $\deg(e)$  of an  $\varepsilon$ -term  $e$  is defined as follows:

1.  $\deg(e) = 1$  iff  $e$  contains no nested  $\varepsilon$ -terms.
2.  $\deg(e) = \max\{\deg(e_1), \dots, \deg(e_n)\} + 1$  if  $e_1, \dots, e_n$  are all the  $\varepsilon$ -terms nested in  $e$ .

For convenience, let  $\deg(t) = 0$  if  $t$  is not an  $\varepsilon$ -term.

**Definition 1.10.** An  $\varepsilon$ -term  $e$  is *subordinate to* an  $\varepsilon$ -term  $e' = \varepsilon_x A(x)$  if some  $e'' \equiv e$  occurs in  $e'$  and  $x \in \text{FV}(e'')$ .

Note that if  $e$  is subordinate to  $e'$  it is *not* a subterm of  $e'$ , because  $x$  is free in  $e$  and so the occurrence of  $e$  (really, of the variant  $e''$ ) in  $e'$  is in the scope of  $\varepsilon_x$ . One might think that replacing  $e$  in  $\varepsilon_x A(x)$  by a new variable  $y$  would result in an  $\varepsilon$ -term  $\varepsilon_x A'(y)$  so that  $e' \equiv \varepsilon_x A'(y)[y/e]$ . But (a)  $\varepsilon_x A'(y)$  is not in general a term, since it is not guaranteed that  $x$  is free in  $A'(y)$  and (b)  $e$  is not free for  $y$  in  $\varepsilon_x A'(y)$ .

**Definition 1.11.** The *rank*  $\text{rk}(e)$  of an  $\varepsilon$ -term  $e$  is defined as follows:

1.  $\text{rk}(e) = 1$  iff  $e$  contains no subordinate  $\varepsilon$ -terms.

2.  $\text{rk}(e) = \max\{\text{rk}(e_1), \dots, \text{rk}(e_n)\} + 1$  if  $e_1, \dots, e_n$  are all the  $\varepsilon$ -terms subordinate to  $e$ .

**Proposition 1.12.** *If  $p$  is the type of  $e$ , then  $\text{rk}(p) = \text{rk}(e)$ .*

*Proof.* Exercise. □

### 1.3 Axioms and Proofs

**Definition 1.13.** The axioms of the *elementary calculus* EC are

$$\begin{array}{lll} A & \text{for any tautology } A & (\text{Taut}) \\ t = t & \text{for any term } t & (=1) \\ t = u \rightarrow (A[x/t] \leftrightarrow A[x/u]) & & (=2) \end{array}$$

and its only rule of inference is

$$\frac{A \quad A \rightarrow B}{A} \text{MP}$$

The axioms and rules of the (intensional)  $\varepsilon$ -calculus  $\text{EC}_\varepsilon$  are those of EC plus the *critical formulas*

$$A(t) \rightarrow A(\varepsilon_x A(x)). \quad (\text{crit})$$

The axioms and rules of the *extensional*  $\varepsilon$ -calculus  $\text{EC}_\varepsilon^{\text{ext}}$  are those of  $\text{EC}_\varepsilon$  plus

$$(\forall x(A(x) \leftrightarrow B(x)))^\varepsilon \rightarrow \varepsilon_x A(x) = \varepsilon_x B(x) \quad (\text{ext})$$

that is,

$$A(\varepsilon_x \neg(A(x) \leftrightarrow B(x))) \leftrightarrow B(\varepsilon_x \neg(A(x) \leftrightarrow B(x))) \rightarrow \varepsilon_x A(x) = \varepsilon_x B(x)$$

The axioms and rules of  $\text{EC}_\forall$ ,  $\text{EC}_{\varepsilon\forall}$ ,  $\text{EC}_{\varepsilon\forall}^{\text{ext}}$  are those of EC,  $\text{EC}_\varepsilon$ ,  $\text{EC}_\varepsilon^{\text{ext}}$ , respectively, together with the axioms

$$\begin{array}{ll} A(t) \rightarrow \exists x A(x) & (\text{Ax}\exists) \\ \forall x A(x) \rightarrow A(t) & (\text{Ax}\forall) \end{array}$$

and the rules

$$\frac{A(x) \rightarrow B}{\exists x A(x) \rightarrow B} R\exists \quad \frac{B \rightarrow A(x)}{B \rightarrow \forall x A(x)} R\forall$$

Applications of these rules must satisfy the *eigenvariable condition*, viz., the variable  $x$  must not appear in the conclusion or anywhere below it in the proof.

**Definition 1.14.** If  $\Gamma$  is a set of formulas, a *proof of  $A$  from  $\Gamma$  in  $\text{EC}_{\varepsilon\forall}^{\text{ext}}$*  is a sequence  $\pi$  of formulas  $A_1, \dots, A_n = A$  where for each  $i \leq n$ , one of the following holds:

1.  $A_i \in \Gamma$ .
2.  $A_i$  is an instance of an axiom.
3.  $A_i$  follows from some  $A_k, A_l$  ( $k, l < i$ ) by (MP), i.e.,  $A_i \equiv C, A_k \equiv B$ , and  $A_l \equiv B \rightarrow C$ .
4.  $A_i$  follows from some  $A_j$  ( $j < i$ ) by (R $\exists$ ), i.e., i.e.,  $A_i \equiv \exists x B(x) \rightarrow C$ ,  $A_j \equiv B(x) \rightarrow C$ , and  $x$  is an eigenvariable, i.e., it satisfies  $x \notin \text{FV}(A_k)$  for any  $k \geq i$  (this includes  $k = i$ , so  $x \notin \text{FV}(C)$ ).
5.  $A_i$  follows from some  $A_j$  ( $j < i$ ) by (R $\forall$ ), i.e., i.e.,  $A_i \equiv C \rightarrow \forall x B(x)$ ,  $A_j \equiv C \rightarrow B(x)$ , and the eigenvariable condition is satisfied.

If  $\pi$  only uses the axioms and rules of EC,  $\text{EC}_\varepsilon$ ,  $\text{EC}_\varepsilon^{\text{ext}}$ , etc., then it is a proof of  $A$  from  $\Gamma$  in EC,  $\text{EC}_\varepsilon$ ,  $\text{EC}_\varepsilon^{\text{ext}}$ , etc., and we write  $\Gamma \vdash^\pi A$ ,  $\Gamma \vdash_\varepsilon^\pi A$ ,  $\Gamma \vdash_{\varepsilon^{\text{ext}}}^\pi A$ , etc.

We say that  $A$  is provable from  $\Gamma$  in EC, etc. ( $\Gamma \vdash A$ , etc.), if there is a proof of  $A$  from  $\Gamma$  in EC, etc.

Note that our definition of proof, because of its use of  $\equiv$ , includes a tacit rule for renaming bound variables. Note also that substitution into members of  $\Gamma$  is *not* permitted. However, we can simulate a provability relation in which substitution into members of  $\Gamma$  is allowed by considering  $\Gamma^{\text{inst}}$ , the set of all substitution instances of members of  $\Gamma$ . If  $\Gamma$  is a set of sentences, then  $\Gamma^{\text{inst}} = \Gamma$ .

**Proposition 1.15.** *If  $\pi = A_1, \dots, A_n \equiv A$  is a proof of  $A$  from  $\Gamma$  and  $x \notin \text{FV}(\Gamma)$  is not an eigenvariable in  $\pi$ , then  $\pi[x/t] = A_1[x/t], \dots, A_n[x/t]$  is a proof of  $A[x/t]$  from  $\Gamma^{\text{inst}}$ .*

*In particular, if  $\Gamma$  is a set of sentences and  $\pi$  is a proof in EC,  $\text{EC}_\varepsilon$ , or  $\text{EC}_\varepsilon^{\text{ext}}$ , then  $\pi[x/t]$  is a proof of  $A[x/t]$  from  $\Gamma$  in EC,  $\text{EC}_\varepsilon$ , or  $\text{EC}_\varepsilon^{\text{ext}}$ .*

*Proof.* Exercise. □

**Lemma 1.16.** *If  $\pi$  is a proof of  $B$  from  $\Gamma \cup \{A\}$ , then there is a proof  $\pi[A]$  of  $A \rightarrow B$  from  $\Gamma$ , provided  $A$  contains no eigenvariables of  $\pi$  free.*

*Proof.* Construct  $\pi[A]_0 = \emptyset$ . Let  $\pi_{i+1}[A] = \pi_i[A]$  plus additional formulas, depending on  $A_i$ :

1. If  $A_i \in \Gamma$ , add  $A \rightarrow A$ , if  $A_i \equiv A$ , or else add  $A_i$ , the tautology  $A_i \rightarrow (A \rightarrow A_i)$ , and  $A \rightarrow A_i$ . The last formula follows from the previous two by (MP).
2. If  $A_i$  is a tautology, add  $A \rightarrow A_i$ , which is also a tautology.
3. If  $A_i$  follows from  $A_k$  and  $A_l$  by (MP), i.e.,  $A_i \equiv C, A_k \equiv B$  and  $A_l \equiv B \rightarrow C$ , then  $\pi[A]_i$  contains  $A \rightarrow B$  and  $A \rightarrow (B \rightarrow C)$ . Add the tautology  $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$  and  $A \rightarrow C$ . The latter follows from the former by two applications of (MP).

4. If  $A_i$  follows from  $A_j$  by  $(R\exists)$ , i.e.,  $A_i \equiv \exists x B(x) \rightarrow C$  and  $A_j \equiv B(x) \rightarrow C$ , then  $\pi[A]_i$  contains  $A \rightarrow (B(x) \rightarrow C)$ .  $\pi[A]_{i+1}$  is

$$\begin{array}{ll}
\pi[A]_i & \\
(A \rightarrow (B(x) \rightarrow C)) \rightarrow (B(x) \rightarrow (A \rightarrow C)) & \text{(taut)} \\
B(x) \rightarrow (A \rightarrow C) & \text{(MP)} \\
\exists x B(x) \rightarrow (A \rightarrow C) & \text{(R}\exists\text{)} \\
(\exists x B(x) \rightarrow (A \rightarrow C)) \rightarrow (A \rightarrow (\exists x B(x) \rightarrow C)) & \text{(taut)} \\
A \rightarrow (\exists x B(x) \rightarrow C) & \text{(MP)}
\end{array}$$

Since  $x \notin \text{FV}(A)$ , the eigenvariable condition is satisfied.

5. Exercise:  $A_i$  follows by  $(R\forall)$ .

Now take  $\pi[A] = \pi[A]_i$ . □

**Theorem 1.17** (Deduction Theorem). *If  $\Sigma \cup \{A\}$  is a set of sentences,  $\Sigma \vdash A \rightarrow B$  iff  $\Sigma \cup \{A\} \vdash B$ .*

**Corollary 1.18.** *If  $\Sigma \cup \{A\}$  is a set of sentences,  $\Sigma \vdash A$  iff  $\Sigma \cup \{\neg A\} \vdash \perp$ .*

**Lemma 1.19** ( $\varepsilon$ -Embedding Lemma). *If  $\Gamma \vdash_{\varepsilon\forall}^{\pi} A$ , then there is a proof  $\pi^\varepsilon$  so that  $\Gamma^{\varepsilon inst} \vdash_{\varepsilon}^{\pi^\varepsilon} A^\varepsilon$*

*Proof.* Exercise. □

## 2 Semantics

### 2.1 Semantics for $\text{EC}_{\varepsilon\forall}^{\text{ext}}$

**Definition 2.1.** A structure  $\mathfrak{M} = \langle |\mathfrak{M}|, (\cdot)^\mathfrak{M} \rangle$  consists of a nonempty domain  $|\mathfrak{M}| \neq \emptyset$  and a mapping  $(\cdot)^\mathfrak{M}$  on function and predicate symbols where:

$$\begin{aligned}
(f_i^0)^\mathfrak{M} &\in |\mathfrak{M}| \\
(f_i^n)^M &\in \mathfrak{M}^{\mathfrak{M}^n} \\
(P_i^n)^\mathfrak{M} &\subseteq \mathfrak{M}^n
\end{aligned}$$

**Definition 2.2.** An extensional choice function  $\Phi$  on  $\mathfrak{M}$  is a function  $\Phi: \wp(|\mathfrak{M}|) \rightarrow |\mathfrak{M}|$  where  $\Phi(X) \in X$  whenever  $X \neq \emptyset$ .

Note that  $\Phi$  is total on  $\wp(|\mathfrak{M}|)$ , and so  $\Phi(\emptyset) \in |\mathfrak{M}|$ .

**Definition 2.3.** An assignment  $s$  on  $\mathfrak{M}$  is a function  $s: \text{Var} \rightarrow |\mathfrak{M}|$ .

If  $x \in \text{Var}$  and  $m \in |\mathfrak{M}|$ ,  $s[x/m]$  is the assignment defined by

$$s[x/m](y) = \begin{cases} m & \text{if } y = x \\ s(y) & \text{otherwise} \end{cases}$$

**Definition 2.4.** The *value*  $\text{val}_{\mathfrak{M}, \Phi, s}(t)$  of a term and the *satisfaction relation*  $\mathfrak{M}, \Phi, s \models A$  are defined as follows:

1.  $\text{val}_{\mathfrak{M}, \Phi, s}(x) = s(x)$
2.  $\mathfrak{M}, \Phi, s \models \top$  and  $\mathfrak{M}, \Phi, s \not\models \perp$
3.  $\text{val}_{\mathfrak{M}, \Phi, s}(f_i^n(t_1, \dots, t_n)) = (f_i^n)^{\mathfrak{M}}(\text{val}_{\mathfrak{M}, \Phi, s}(t_1), \dots, \text{val}_{\mathfrak{M}, \Phi, s}(t_n))$
4.  $\mathfrak{M}, \Phi, s \models P_i^n(t_1, \dots, t_n)$  iff  $\langle \text{val}_{\mathfrak{M}, \Phi, s}(t_1), \dots, \text{val}_{\mathfrak{M}, \Phi, s}(t_n) \rangle \in (P_i^n)^{\mathfrak{M}}$
5.  $\text{val}_{\mathfrak{M}, \Phi, s}(\varepsilon_x A(x)) = \Phi(\text{val}_{\mathfrak{M}, \Phi, s}(A(x)))$  where

$$\text{val}_{\mathfrak{M}, \Phi, s}(A(x)) = \{m \in |\mathfrak{M}| : \mathfrak{M}, \Phi, s[x/m] \models A(x)\}$$

6.  $\mathfrak{M}, \Phi, s \models \exists x A(x)$  iff for some  $m \in |\mathfrak{M}|$ ,  $\mathfrak{M}, \Phi, s[x/m] \models A(x)$
7.  $\mathfrak{M}, \Phi, s \models \forall x A(x)$  iff for all  $m \in |\mathfrak{M}|$ ,  $\mathfrak{M}, \Phi, s[x/m] \models A(x)$

**Proposition 2.5.** If  $s(x) = s'(x)$  for all  $x \notin \text{FV}(t) \cup \text{FV}(A)$ , then  $\text{val}_{\mathfrak{M}, \Phi, s}(t) = \text{val}_{\mathfrak{M}, \Phi, s'}(t)$  and  $\mathfrak{M}, \Phi, s \models A$  iff  $\mathfrak{M}, \Phi, s' \models A$ .

*Proof.* Exercise. □

**Proposition 2.6** (Substitution Lemma). If  $m = \text{val}_{\mathfrak{M}, \Phi, s}(u)$ , then  $\text{val}_{\mathfrak{M}, \Phi, s}(t(u)) = \text{val}_{\mathfrak{M}, \Phi, s[x/m]}(t(x))$  and  $\mathfrak{M}, \Phi, s \models A(u)$  iff  $\mathfrak{M}, \Phi, s[x/m] \models A(x)$

*Proof.* Exercise. □

**Definition 2.7.** 1.  $A$  is *locally true* in  $\mathfrak{M}$  with respect to  $\Phi$  and  $s$  iff  $\mathfrak{M}, \Phi, s \models A$ .

2.  $A$  is *true* in  $\mathfrak{M}$  with respect to  $\Phi$ ,  $\mathfrak{M}, \Phi \models A$ , iff for all  $s$  on  $\mathfrak{M}$ :  $\mathfrak{M}, \Phi, s \models A$ .
3.  $A$  is *generically true* in  $\mathfrak{M}$  with respect to  $s$ ,  $\mathfrak{M}, s \models^g A$ , iff for all choice functions  $\Phi$  on  $\mathfrak{M}$ :  $\mathfrak{M}, \Phi, s \models A$ .
4.  $A$  is *generically valid* in  $\mathfrak{M}$ ,  $\mathfrak{M} \models A$ , if for all choice functions  $\Phi$  and assignments  $s$  on  $\mathfrak{M}$ :  $\mathfrak{M}, \Phi, s \models A$ .

**Definition 2.8.** Let  $\Gamma \cup \{A\}$  be a set of formulas.

1.  $A$  is a *local consequence* of  $\Gamma$ ,  $\Gamma \models^l A$ , iff for all  $\mathfrak{M}, \Phi$ , and  $s$ :  
if  $\mathfrak{M}, \Phi, s \models \Gamma$  then  $\mathfrak{M}, \Phi, s \models A$ .
2.  $A$  is a *truth consequence* of  $\Gamma$ ,  $\Gamma \models A$ , iff for all  $\mathfrak{M}, \Phi$ :  
if  $\mathfrak{M}, \Phi \models \Gamma$  then  $\mathfrak{M}, \Phi \models A$ .
3.  $A$  is a *generic consequence* of  $\Gamma$ ,  $\Gamma \models^g A$ , iff for all  $\mathfrak{M}$  and  $s$ :  
if  $\mathfrak{M}, s \models^g \Gamma$  then  $\mathfrak{M} \models A$ .



4.  $A$  is a *generic validity consequence* of  $\Gamma$ ,  $\Gamma \models^v A$ , iff for all  $\mathfrak{M}$ :  
if  $\mathfrak{M} \models^v \Gamma$  then  $\mathfrak{M} \models A$ .

**Exercise 2.** What is the relationship between these consequence relations? For instance, if  $\Gamma \models^l A$  then  $\Gamma \models A$  and  $\Gamma \models^g A$ , and if either  $\Gamma \models A$  or  $\Gamma \models^g A$ , then  $\Gamma \models^v A$ . Are these containments strict? Are they identities (in general, and in cases where the language of  $\Gamma, A$  is restricted, or if  $\Gamma, A$  are sentences)? For instance:

**Proposition 2.9.** If  $\Sigma \cup \{A\}$  is a set of sentences,  $\Sigma \models^l A$  iff  $\Sigma \models A$

**Proposition 2.10.** If  $\Sigma \cup \{A, B\}$  is a set of sentences,  $\Sigma \cup \{A\} \models B$  iff  $\Sigma \models A \rightarrow B$ .

*Proof.* Exercise. □

**Corollary 2.11.** If  $\Sigma \cup \{A\}$  is a set of sentences,  $\Sigma \models A$  iff for no  $\mathfrak{M}, \Phi, \mathfrak{M} \models \Sigma \cup \{\neg A\}$

*Proof.* Exercise. □

**Exercise 3.** For which of the other consequence relations, if any, do these results hold?

## 2.2 Soundness for $\text{EC}_{\varepsilon\forall}^{\text{ext}}$

**Theorem 2.12.** If  $\Gamma \vdash_{\varepsilon\forall} A$ , then  $\Gamma \models^l A$ .

*Proof.* Suppose  $\Gamma, \Phi, s \models \Gamma$ . We show by induction on the length  $n$  of a proof  $\pi$  that  $\mathfrak{M}, \Phi, s \models^l A$  for all  $s'$  which agree with  $s$  on  $\text{FV}(\Gamma)$ . We may assume that no eigenvariable  $x$  of  $\pi$  is in  $\text{FV}(\Gamma)$  (if it is, let  $y \notin \text{FV}(\pi)$  and not occurring in  $\pi$ ; consider  $\pi[x/y]$  instead of  $\pi$ ).

If  $n = 0$  there's nothing to prove. Otherwise, we distinguish cases according to the last line  $A_n$  in  $\pi$ :

1.  $A_n \in \Gamma$ . The claim holds by assumption.
2.  $A_n$  is a tautology. Obvious.
3.  $A_n$  is an identity axiom. Obvious.
4.  $A_n$  is a critical formula, i.e.,  $A_n \equiv A(t) \rightarrow A(\varepsilon_x A(x))$ . Then either  $\mathfrak{M}, \Phi, s \models A(t)$  or not (in which case there's nothing to prove). If yes,  $\mathfrak{M}, \Phi, s[x/m] \models A(x)$  for  $m = \text{val}_{\mathfrak{M}, \Phi, s}(t)$ , and so  $Y = \text{val}_{\mathfrak{M}, \Phi, s}(A(x)) \neq \emptyset$ . Consequently,  $\Phi(Y) \in Y$ , and hence  $\mathfrak{M}, \Phi, s \models A(\varepsilon_x A(x))$ .
5.  $A_n$  is an extensionality axiom. Exercise.
6.  $A_n$  follows from  $B$  and  $B \rightarrow C$  by (MP). By induction hypothesis,  $\mathfrak{M}, \Phi, s \models B$  and  $\mathfrak{M}, \Phi, s \models B \rightarrow C$ .

7.  $A$  follows from  $B(x) \rightarrow C$  by  $(R\exists)$ , and  $x$  satisfies the eigenvariable condition. Exercise.
8.  $A$  follows from  $C \rightarrow B(x)$  by  $(R\forall)$ , and  $x$  satisfies the eigenvariable condition. Exercise.

□

**Exercise 4.** Complete the missing cases.

### 2.3 Completeness for $EC_{\varepsilon\forall}^{\text{ext}}$

**Lemma 2.13.** *If  $\Gamma$  is a set of sentences in  $L_\varepsilon$  and  $\Gamma \not\vdash_\varepsilon \perp$ , then there are  $\mathfrak{M}, \Phi$  so that  $\mathfrak{M}, \Phi \models \Gamma$ .*

**Theorem 2.14** (Completeness). *If  $\Gamma \cup \{A\}$  are sentences in  $L_\varepsilon$  and  $\Gamma \models A$ , then  $\Gamma \vdash_\varepsilon A$ .*

*Proof.* Suppose  $\Gamma \not\models A$ . Then for some  $\mathfrak{M}, \Phi$  we have  $\mathfrak{M}, \Phi \models \Gamma$  but  $\mathfrak{M}, \Phi \not\models A$ . Hence  $\mathfrak{M}, \Phi \models \Gamma \cup \{\neg A\}$ . By the Lemma,  $\Gamma \cup \{\neg A\} \vdash_\varepsilon \perp$ . By Corollary 1.18,  $\Gamma \vdash_\varepsilon A$ . □

The proof of the Lemma comes in several stages. We have to show that if  $\Gamma$  is consistent, we can construct  $\mathfrak{M}, \Phi$ , and  $s$  so that  $\mathfrak{M}, \Phi, s \models \Gamma$ . Since  $FV(\Gamma) = \emptyset$ , we then have  $\mathfrak{M}, \Phi \models \Gamma$ .

**Lemma 2.15.** *If  $\Gamma \not\vdash_\varepsilon \perp$ , there is  $\Gamma^* \supseteq \Gamma$  with (1)  $\Gamma^* \not\vdash_\varepsilon \perp$  and (2) for all formulas  $A$ , either  $A \in \Gamma^*$  or  $\neg A \in \Gamma^*$ .*

*Proof.* Let  $A_1, A_2, \dots$  be an enumeration of  $\text{Frm}_\varepsilon$ . Define  $\Gamma_0 = \Gamma$  and

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{A_n\} & \text{if } \Gamma_n \cup \{A_n\} \not\vdash_\varepsilon \perp \\ \Gamma_n \cup \{\neg A_n\} & \text{if } \Gamma_n \cup \{\neg A_n\} \not\vdash_\varepsilon \perp \text{ otherwise} \end{cases}$$

Let  $\Gamma^* = \bigcup_{n \geq 0} \Gamma_n$ . Obviously,  $\Gamma \subseteq \Gamma^*$ . For (1), observe that if  $\Gamma^* \vdash_\varepsilon^\pi \perp$ , then  $\pi$  contains only finitely many formulas from  $\Gamma^*$ , so for some  $n$ ,  $\Gamma_n \vdash_\varepsilon^\pi \perp$ . But  $\Gamma_n$  is consistent by definition.

To verify (2), we have to show that for each  $n$ , either  $\Gamma_n \cup \{A_n\} \not\vdash_\varepsilon \perp$  or  $\Gamma_n \cup \{\neg A_n\} \not\vdash_\varepsilon \perp$ . For  $n = 0$ , this is the assumption of the lemma. So suppose the claim holds for  $n - 1$ . Suppose  $\Gamma_n \cup \{A\} \vdash_\varepsilon^\pi \perp$  and  $\Gamma_n \cup \{\neg A\} \vdash_\varepsilon^{\pi'} \perp$ . Then by the Deduction Theorem, we have  $\Gamma_n \vdash_\varepsilon^{\pi[A]} A \rightarrow \perp$  and  $\Gamma_n \vdash_\varepsilon^{\pi'[A']} A \rightarrow \perp$ . Since  $(A \rightarrow \perp) \rightarrow ((\neg A \rightarrow \perp) \rightarrow \perp)$  is a tautology, we have  $\Gamma_n \vdash_\varepsilon \perp$ , contradicting the induction hypothesis. □

**Lemma 2.16.** *If  $\Gamma^* \vdash_\varepsilon B$ , then  $B \in \Gamma^*$ .*

*Proof.* If not, then  $\neg B \in \Gamma^*$  by maximality, so  $\Gamma^*$  would be inconsistent. □

**Definition 2.17.** Let  $\approx$  be the relation on  $\text{Trm}_\varepsilon$  defined by

$$t \approx u \text{ iff } t = u \in \Gamma^*$$

It is easily seen that  $\approx$  is an equivalence relation. Let  $\tilde{t} = \{u : u \approx t\}$  and  $\widetilde{\text{Trm}} = \{\tilde{t} : t \in \text{Trm}\}$ .

**Definition 2.18.** A set  $T \in \widetilde{\text{Trm}}$  is *represented by*  $A(x)$  if  $T = \{\tilde{t} : A(t) \in \Gamma^*\}$ .

Let  $\Phi_0$  be a fixed choice function on  $\widetilde{\text{Trm}}$ , and define

$$\Phi(T) = \begin{cases} \varepsilon_x A(x) & \text{if } T \text{ is represented by } A(x) \\ \Phi_0(T) & \text{otherwise.} \end{cases}$$

**Proposition 2.19.**  $\Phi$  is a well-defined choice function on  $\widetilde{\text{Trm}}$ .

*Proof.* Exercise. Use (ext) for well-definedness and (crit) for choice function.  $\square$

Now let  $\mathfrak{M} = \langle \widetilde{\text{Trm}}, (\cdot)^\mathfrak{M} \rangle$  with  $c^\mathfrak{M} = \tilde{c}$ ,  $(P_i^n)^\mathfrak{M} = \{\langle \tilde{t}_1, \dots, \tilde{t}_1 \rangle : P_i^n(t_1, \dots, t_n)\}$ , and let  $s(x) = \tilde{s}$ .

**Proposition 2.20.**  $\mathfrak{M}, \Phi, s \models \Gamma^*$ .

*Proof.* We show that  $\text{val}_{\mathfrak{M}, \Phi, s}(t) = \tilde{t}$  and  $\mathfrak{M}, \Phi, s \models A$  iff  $A \in \Gamma^*$  by simultaneous induction on the complexity of  $t$  and  $A$ .

If  $t = c$  is a constant, the claim holds by definition of  $(\cdot)^\mathfrak{M}$ . If  $A = \perp$  or  $A = \top$ , the claim holds by Lemma 2.16.

If  $A \equiv P^n(t_1, \dots, t_n)$ , then by induction hypothesis,  $\text{val}_{\mathfrak{M}, \Phi, s}(t)_i = \tilde{t}_i$ . By definition of  $(\cdot)^\mathfrak{M}$ ,  $\langle \tilde{t}_1, \dots, \tilde{t}_n \rangle \in (P_i^n)^\mathfrak{M}(t_1, \dots, t_n)$  iff  $P_i^n(t_1, \dots, t_n) \in \Gamma^*$ .

If  $A \equiv \neg B$ ,  $(B \wedge C)$ ,  $(B \vee C)$ ,  $(B \rightarrow C)$ ,  $(B \leftrightarrow C)$ , the claim follows immediately from the induction hypothesis and the definition of  $\models$  and the closure properties of  $\Gamma^*$ . For instance,  $\mathfrak{M}, \Phi, s \models (B \wedge C)$  iff  $\mathfrak{M}, \Phi, s \models B$  and  $\mathfrak{M}, \Phi, s \models C$ . By induction hypothesis, this is the case iff  $B \in \Gamma^*$  and  $C \in \Gamma^*$ . But since  $B, C \vdash_\varepsilon B \wedge C$  and  $B \wedge C \vdash_\varepsilon B$  and  $\vdash_\varepsilon C$ , this is the case iff  $(B \wedge C) \in \Gamma^*$ . Remaining cases: Exercise.

If  $t \equiv \varepsilon_x A(x)$ , then  $\text{val}_{\mathfrak{M}, \Phi, s}(t) = \Phi(\text{val}_{\mathfrak{M}, \Phi, s}(A(x)))$ . Since  $\text{val}_{\mathfrak{M}, \Phi, s}(A(x))$  is represented by  $A(x)$  by induction hypothesis, we have  $\text{val}_{\mathfrak{M}, \Phi, s}(t) = \varepsilon_x A(x)$  by definition of  $\Phi$ .  $\square$

**Exercise 5.** Complete the proof.

**Exercise 6.** Generalize the proof to  $L_{\varepsilon\forall}$  and  $\text{EC}_{\varepsilon\forall}$ .

**Exercise 7.** Show  $\text{EC}_\varepsilon$  without  $(=_1)$  and  $(=_2)$ , (ext), and the additional axiom

$$(\forall x(A(x) \leftrightarrow B(x)))^\varepsilon \rightarrow (C(\varepsilon_x A(x)) \leftrightarrow C(\varepsilon_x B(x))) \quad (\text{ext}^-)$$

is complete for  $\models$  in the language  $L_{\varepsilon\forall}^-$ .

## 2.4 Semantics for $EC_{\varepsilon\forall}$

In order to give a complete semantics for  $EC_{\varepsilon\forall}$ , i.e., for the calculus without the extensionality axiom (ext), it is necessary to change the notion of choice function so that two  $\varepsilon$ -terms  $\varepsilon_x A(x)$  and  $\varepsilon_x B(x)$  may be assigned different representatives even when  $\mathfrak{M}, \Phi, s \models \forall x (A(x) \leftrightarrow B(x))$ , since then the negation of (ext) is consistent in the resulting calculus. The idea is to add the  $\varepsilon$ -term itself as an additional argument to the choice function. However, in order for this semantics to be sound for the calculus—specifically, in order for  $(=_2)$  to be valid—we have to use not  $\varepsilon$ -terms but  $\varepsilon$ -types.

**Definition 2.21.** An *intensional choice operator* is a mapping  $\Psi: \text{Typ} \times |\mathfrak{M}|^{<\omega} \rightarrow |\mathfrak{M}|^{\wp(|\mathfrak{M}|)}$  such that for every type  $p = \varepsilon_x A(x; y_1, \dots, y_n)$  is a type, and  $m_1, \dots, m_n \in |\mathfrak{M}|$ ,  $\Psi(p, m_1, \dots, m_n)$  is a choice function.

**Definition 2.22.** If  $\mathfrak{M}$  is a structure,  $\Psi$  an intensional choice operator, and  $s$  an assignment,  $\text{val}_{\mathfrak{M}, \Psi, s}(t)$  and  $\mathfrak{M}, \Psi, s \models A$  is defined as before, except (5) in Definition 2.4 is replaced by:

- (5')  $\text{val}_{\mathfrak{M}, \Psi, s}(\varepsilon_x A(x)) = \Psi(p, m_1, \dots, m_n)(\text{val}_{\mathfrak{M}, \Phi, s}(A(x)))$  where
- a)  $p = \varepsilon_x A'(x; x_1, \dots, x_n)$  is the type of  $\varepsilon_x A(x)$ ,
  - b)  $t_1, \dots, t_n$  are the subterms corresponding to  $x_1, \dots, x_n$ , i.e.,  $\varepsilon_x A(x) \equiv \varepsilon_x A'(x; t_1, \dots, t_n)$ ,
  - c)  $m_i = \text{val}_{\mathfrak{M}, \Psi, s}(t_i)$ , and
  - d)  $\text{val}_{\mathfrak{M}, \Phi, s}(A(x)) = \{m \in |\mathfrak{M}| : \mathfrak{M}, \Psi, s[x/m] \models A(x)\}$

**Exercise 8.** Prove the substitution lemma for this semantics.

**Exercise 9.** Prove soundness.

**Exercise 10.** Prove completeness of  $EC_{\varepsilon\forall}$  for this semantics.

**Exercise 11.** Define a semantics for the language without  $=$  where the choice operator takes  $\varepsilon$ -terms as arguments. Is the semantics sound and complete for  $EC_{\varepsilon\forall}^-$ ?

## 3 The First Epsilon Theorem

### 3.1 The Case Without Identity

**Theorem 3.1.** If  $E$  is a formula not containing any  $\varepsilon$ -terms and  $\vdash_{EC_{\varepsilon\forall}} E$ , then  $\vdash_{EC} E$ .

**Definition 3.2.** An  $\varepsilon$ -term  $e$  is *critical* in  $\pi$  if  $A(t) \rightarrow A(e)$  is one of the critical formulas in  $\pi$ . The *rank*  $\text{rk}(\pi)$  of a proof  $\pi$  is the maximal rank of its critical  $\varepsilon$ -terms. The *r-degree*  $\text{deg}(\pi, r)$  of  $\pi$  is the maximum degree of its critical  $\varepsilon$ -terms of rank  $r$ . The *r-order*  $o(\pi, r)$  of  $\pi$  is the number of different (up to renaming of bound variables) critical  $\varepsilon$ -terms of rank  $r$ .

**Lemma 3.3.** *If  $e = \varepsilon_x A(x), \varepsilon_y B(y)$  are critical in  $\pi$ ,  $\text{rk}(e) = \text{rk}(\pi)$ , and  $B^* \equiv B(u) \rightarrow B(\varepsilon_y B(y))$  is a critical formula in  $\pi$ . Then, if  $e$  is a subterm of  $B^*$ , it is a subterm of  $B(y)$  or a subterm of  $u$ .*

*Proof.* Suppose not. Then, since  $e$  is a subterm of  $B^*$ , we have  $B(y) \equiv B'(\varepsilon_x A'(x, y), y)$  and either  $e \equiv \varepsilon_x A'(x, u)$  or  $e \equiv \varepsilon_x A'(x, \varepsilon_y B(y))$ . In each case, we see that  $\varepsilon_x A'(x, y)$  and  $e$  have the same rank, since the latter is an instance of the former (and so have the same type). On the other hand, in either case,  $\varepsilon_y B(y)$  would be

$$\varepsilon_y B'(\varepsilon_x A'(x, y), y)$$

and so would have a higher rank than  $\varepsilon_x A'(x, y)$  as that  $\varepsilon$ -term is subordinate to it. This contradicts  $\text{rk}(e) = \text{rk}(\pi)$ .  $\square$

**Lemma 3.4.** *Let  $e, B^*$  be as in the lemma, and  $t$  be any term. Then*

1. *If  $e$  is not a subterm of  $B(y)$ ,  $B^*\{e/t\} \equiv B(u') \rightarrow B(\varepsilon_y B(y))$ .*
2. *If  $e$  is a subterm of  $B(y)$ , i.e.,  $B(y) \equiv B'(e, y)$ ,  $B^*\{e/t\} \equiv B'(t, u') \rightarrow B'(t, \varepsilon_y B'(t, y))$ .*

*Proof.* By inspection.  $\square$

**Lemma 3.5.** *If  $\vdash_{\text{EC}_\varepsilon}^\pi E$  and  $E$  does not contain  $\varepsilon$ , then there is a proof  $\pi'$  such that  $\vdash_{\text{EC}_\varepsilon}^{\pi'} E$  and  $\text{rk}(\pi') \leq \text{rk}(\pi) = r$  and  $o(\pi', r) < o(\pi, r)$ .*

*Proof.* Let  $e$  be an  $\varepsilon$ -term critical in  $\pi$  and let  $A(t_1) \rightarrow A(e)$ , dots,  $A(t_n) \rightarrow A(e)$  be all its critical formulas in  $\pi$ .

Consider  $\pi\{e/t\}_i$ , i.e.,  $\pi$  with  $e$  replaced by  $t_i$  throughout. Each critical formula belonging to  $e$  now is of the form  $A(t'_j) \rightarrow A(t_i)$ , since  $e$  obviously cannot be a subterm of  $A(x)$  (if it were,  $e$  would be a subterm of  $\varepsilon_x A(x)$ , i.e., of itself!). Let  $\hat{\pi}_i$  be the sequence of tautologies  $A(t_i) \rightarrow (A(t'_j) \rightarrow A(t_i))$  for  $i = 1, \dots, n$ , followed by  $\pi\{e/t\}_i$ . Each one of the formulas  $A(t'_j) \rightarrow A(t_i)$  follows from one of these by (MP) from  $A(t_i)$ . Hence,  $A(t_i) \vdash_{\text{EC}_\varepsilon}^{\hat{\pi}_i} E$ . Let  $\pi_i = \hat{\pi}_i[A_i]$  as in Lemma 1.16. We have  $\vdash_{\text{EC}_\varepsilon}^{\pi_i} A_i \rightarrow E$ .

The  $\varepsilon$ -term  $e$  is not critical in  $\pi_i$ : Its original critical formulas are replaced by  $A(t_i) \rightarrow (A(t'_j) \rightarrow A(t_i))$ , which are tautologies. By (1) of the preceding Lemma, no critical  $\varepsilon$ -term of rank  $r$  was changed at all. By (2) of the preceding Lemma, no critical  $\varepsilon$ -term of rank  $< r$  was replaced by a critical  $\varepsilon$ -term of rank  $\geq r$ . Hence,  $o(\pi_i, r) = o(\pi) - 1$ .

Let  $\pi''$  be the sequence of tautologies  $\neg \bigvee_{i=1}^n A(t_i) \rightarrow (A(t_i) \rightarrow A(e))$  followed by  $\pi$ . Then  $\bigvee_{i=1}^n A(t_i) \vdash_E^{\pi''} E$ ,  $e$  is not critical in  $\pi''$ , and otherwise  $\pi$  and  $\pi''$  have the same critical formulas. The same goes for  $\pi''[\neg \bigvee A(t_i)]$ , a proof of  $\neg \bigvee A(t_i) \rightarrow E$ .

We now obtain  $\pi'$  as the  $\pi_i, i = 1, \dots, n$ , followed by  $\pi[\neg \bigvee_{i=1}^n]$ , followed by the tautology

$$(\neg \bigvee A(t_i) \rightarrow E) \rightarrow (A(t_1) \rightarrow E) \rightarrow \dots \rightarrow (A(t_n) \rightarrow E) \rightarrow E) \dots)$$

from which  $E$  follows by  $n + 1$  applications of (MP).  $\square$

of the first  $\varepsilon$ -Theorem. By induction on  $o(\pi, r)$ , we have: if  $\vdash_{\text{EC}_\varepsilon}^\pi E$ , then there is a proof  $\pi^*$  of  $E$  with  $\text{rk}(\pi^-) < r$ . By induction on  $\text{rk}(\pi)$  we have a proof  $\pi^{**}$  of  $E$  with  $\text{rk}(\pi^{**}) = 0$ , i.e., without critical formulas at all.  $\square$

**Exercise 12.** Check these proofs. Can you think of ways to improve the proofs?

**Exercise 13.** If  $E$  contains  $\varepsilon$ -terms, the replacement of  $\varepsilon$ -terms in the construction of  $\pi_i$  may change  $E$ —but of course only the  $\varepsilon$ -terms appearing as subterms in it. Use this fact to prove: If  $\vdash_{\text{EC}_{\varepsilon\forall}} E(e)$ , then  $\vdash_{\text{EC}} \bigvee_{i=1}^m E(t_i)$  for some terms  $t_j$ . Can you guarantee that  $t_j$  are  $\varepsilon$ -free.

### 3.2 The Case With Identity

In the presence of the identity ( $=$ ) predicate in the language, things get a bit more complicated. The reason is that instances of the ( $=_2$ ) axiom schema,

$$t = u \rightarrow (A(t) \rightarrow A(u))$$

may also contain  $\varepsilon$ -terms, and the replacement of an  $\varepsilon$ -term  $e$  by a term  $t_i$  in the construction of  $\pi_i$  may result in a formula which no longer is an instance of ( $=_2$ ). For instance, suppose that  $t$  is a subterm of  $e = e'(t)$  and  $A(t)$  is of the form  $A'(e'(t))$ . Then the original axiom is

$$t = u \rightarrow (A'(e'(t)) \rightarrow A'(e'(u)))$$

which after replacing  $e = e'(t)$  by  $t_i$  turns into

$$t = u \rightarrow (A'(t_i) \rightarrow A'(e'(u))).$$

So this must be avoided. In order to do this, we first observe that just as in the case of the predicate calculus, the instances of ( $=_2$ ) can be derived from restricted instances. In the case of the predicate calculus, the restricted axioms are

$$t = u \rightarrow (P^n(s_1, \dots, t, \dots, s_n) \rightarrow P^n(s_1, \dots, u, \dots, s_n)) \quad (='_2)$$

$$t = u \rightarrow f^n(s_1, \dots, t, \dots, s_n) = f^n(s_1, \dots, u, \dots, s_n) \quad (=''_2)$$

to which we have to add the  $\varepsilon$ -identity axiom schema:

$$t = u \rightarrow \varepsilon_x A(x; s_1, \dots, t, \dots, s_n) = \varepsilon_x A(x; s_1, \dots, u, \dots, s_n) \quad (=_\varepsilon)$$

where  $\varepsilon_x A(x; x_1, \dots, x_n)$  is an  $\varepsilon$ -type.

**Proposition 3.6.** *Every instance of  $(=_2)$  can be derived from  $(='_2)$ ,  $(=''_2)$ , and  $(=_\varepsilon)$ .*

*Proof.* Exercise. □

Now replacing every occurrence of  $e$  in an instance of  $(='_2)$  or  $(=''_2)$ —where  $e$  obviously can only occur inside one of the terms  $t, u, s_1, \dots, s_n$ —results in a (different) instance of  $(='_2)$  or  $(=''_2)$ . The same is true of  $(=_\varepsilon)$ , *provided that* the  $e$  is neither  $\varepsilon_x A(x; s_1, \dots, t, \dots, s_n)$  nor  $\varepsilon_x A(x; s_1, \dots, u, \dots, s_n)$ . This would be guaranteed if the type of  $e$  is not  $\varepsilon_x A(x; x_1, \dots, x_n)$ , in particular, if the rank of  $e$  is higher than the rank of  $\varepsilon_x A(x; x_1, \dots, x_n)$ . Moreover, the result of replacing  $e$  by  $t_i$  in any such instance of  $(=_\varepsilon)$  results in an instance of  $(=_\varepsilon)$  which belongs to the same  $\varepsilon$ -type. Thus, in order for the proof of the first  $\varepsilon$ -theorem to work also when  $=$  and axioms  $(=_1)$ ,  $(='_2)$ ,  $(=''_2)$ , and  $(=_\varepsilon)$  are present, it suffices to show that the instances of  $(=_\varepsilon)$  with  $\varepsilon$ -terms of rank  $\text{rk}(\pi)$  can be removed. Call an  $\varepsilon$ -term  $e$  *special* in  $\pi$ , if  $\pi$  contains an occurrence of  $t = u \rightarrow e' = e$  as an instance of  $(=_\varepsilon)$ .

**Theorem 3.7.** *If  $\vdash_{\text{EC}_\varepsilon}^\pi E$ , then there is a proof  $\pi^-$  so that  $\vdash_{\text{EC}_\varepsilon}^{\pi^-} E$ ,  $\text{rk}(\pi^-) = \text{rk}(\pi)$ , and the rank of the special  $\varepsilon$ -terms in  $\pi^-$  has rank  $< \text{rk}(\pi)$ .*

The basic idea is simple: Suppose  $t = u \rightarrow e' = e$  is an instance of  $(=_\varepsilon)$ , with  $e' \equiv \varepsilon_x A(x; s_1, \dots, t, \dots, s_n)$  and  $e \equiv \varepsilon_x A(x; s_1, \dots, u, \dots, s_n)$ . Replace  $e$  everywhere in the proof by  $e'$ . Then the instance of  $(=_\varepsilon)$  under consideration is removed, since it is now provable from  $e' = e'$ . This potentially interferes with critical formulas belonging to  $e$ , but this can also be fixed: we just have to show that by a judicious choice of  $e$  it can be done in such a way that the other  $(=_\varepsilon)$  axioms are still of the required form.

Let  $p = \varepsilon_x A(x; x_1, \dots, x_n)$  be an  $\varepsilon$ -type of rank  $\text{rk}(\pi)$ , and let  $e_1, \dots, e_l$  be all the  $\varepsilon$ -terms of type  $p$  which have a corresponding instance of  $(=_\varepsilon)$  in  $\pi$ . Let  $T_i$  be the set of all immediate subterms of  $e_1, \dots, e_l$ , in the same position as  $x_i$ , i.e., the smallest set of terms so that if  $e_i \equiv \varepsilon_x A(x; t_1, \dots, t_n)$ , then  $t_i \in T$ . Now let  $T^*$  be all instances of  $p$  with terms from  $T_i$  substituted for the  $x_i$ . Obviously,  $T$  and thus  $T^*$  are finite (up to renaming of bound variables). Pick a strict order  $\prec$  on  $T$  which respects degree, i.e., if  $\deg(t) < \deg(u)$  then  $t \prec u$ . Extend  $\prec$  to  $T^*$  by

$$\varepsilon_x A(x; t_1, \dots, t_n) \prec \varepsilon_x A(x; t'_1, \dots, t'_n)$$

iff

1.  $\max\{\deg(t_i) : i = 1, \dots, n\} < \max\{\deg(t'_i) : i = 1, \dots, n\}$  or
2.  $\max\{\deg(t_i) : i = 1, \dots, n\} = \max\{\deg(t'_i) : i = 1, \dots, n\}$  and
  - a)  $t_i \equiv t'_i$  for  $i = 1, \dots, k$ .
  - b)  $t_{k+1} \prec t'_{k+1}$

**Lemma 3.8.** Suppose  $\vdash_{\text{EC}_\varepsilon}^\pi E$ ,  $e$  a special  $\varepsilon$ -term in  $\pi$  with  $\text{rk}(e) = \text{rk}(\pi)$ ,  $\deg(e)$  maximal among the special  $\varepsilon$ -terms of rank  $\text{rk}(\pi)$ , and  $e$  maximal with respect to  $\prec$  defined above. Let  $t = u \rightarrow e' = e$  be an instance of  $(=_\varepsilon)$  in  $\pi$ . Then there is a proof  $\pi', \vdash_{\text{EC}_\varepsilon}^{\pi'} E$  such that

1.  $\text{rk}(\pi') = \text{rk}(\pi)$
2.  $\pi'$  does not contain  $t = u \rightarrow e' = e$  as an axiom
3. Every special  $\varepsilon$ -term  $e''$  of  $\pi'$  with the same type as  $e$  is so that  $e'' \prec e$ .

*Proof.* Let  $\pi_0 = \pi\{e/e'\}$ .

Suppose  $t' = u' \rightarrow e''' = e''$  is an  $(=_\varepsilon)$  axiom in  $\pi$ .

If  $\text{rk}(e'') < \text{rk}(e)$ , then the replacement of  $e$  by  $e'$  can only change subterms of  $e''$  and  $e'''$ . In this case, the uniform replacement results in another instance of  $(=_\varepsilon)$  with  $\varepsilon$ -terms of the same  $\varepsilon$ -type, and hence of the same rank  $< \text{rk}(\pi)$ , as the original.

If  $\text{rk}(e'') = \text{rk}(e)$  but has a different type than  $e$ , then this axiom is unchanged in  $\pi_0$ : Neither  $e''$  nor  $e'''$  can be  $\equiv e$ , because they have different  $\varepsilon$ -types, and neither  $e''$  nor  $e'''$  (nor  $t'$  or  $u'$ , which are subterms of  $e''$ ,  $e'''$ ) can contain  $e$  as a subterm, since then  $e$  wouldn't be degree-maximal among the special  $\varepsilon$ -terms of  $\pi$  of rank  $\text{rk}(\pi)$ .

If the type of  $e''$ ,  $e'''$  is the same as that of  $e$ ,  $e$  cannot be a proper subterm of  $e''$  or  $e'''$ , since otherwise  $e''$  or  $e'''$  would again be a special  $\varepsilon$ -term of rank  $\text{rk}(\pi)$  but of higher degree than  $e$ . So either  $e \equiv e''$  or  $e \equiv e'''$ , without loss of generality suppose  $e \equiv e''$ . Then the  $(=_\varepsilon)$  axiom in question has the form

$$t' = u' \rightarrow \underbrace{\varepsilon_x A(x; s_1, \dots, t', \dots, s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots, u', \dots, s_n)}_{e'' \equiv e}$$

and with  $e$  replaced by  $e'$ :

$$t' = u' \rightarrow \underbrace{\varepsilon_x A(x; s_1, \dots, t', \dots, s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots, t, \dots, s_n)}_{e'}$$

which is no longer an instance of  $(=_\varepsilon)$ , but can be proved from new instances of  $(=_\varepsilon)$ . We have to distinguish two cases according to whether the indicated position of  $t$  and  $t'$  in  $e'$ ,  $e'''$  is the same or not. In the first case,  $u \equiv u'$ , and the new formula

$$t' = u \rightarrow \underbrace{\varepsilon_x A(x; s_1, \dots, t', \dots, s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots, t, \dots, s_n)}_{e'}$$

can be proved from  $t = u$  together with

$$t' = t \rightarrow \underbrace{\varepsilon_x A(x; s_1, \dots, t', \dots, s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots, t, \dots, s_n)}_{e'} \quad (=_\varepsilon)$$

$$t = u \rightarrow (t' = u \rightarrow t' = t) \quad (='_2)$$



Since  $e'$  and  $e'''$  already occurred in  $\pi$ , by assumption  $e', e''' \prec e$ .

In the second case, the original formulas read, with terms indicated:

$$\begin{aligned} t = u &\rightarrow \underbrace{\varepsilon_x A(x; s_1, \dots, t, \dots, u', \dots, s_n)}_{e'} = \underbrace{\varepsilon_x A(x; s_1, \dots, u, \dots, u', \dots, s_n)}_e \\ t' = u' &\rightarrow \underbrace{\varepsilon_x A(x; s_1, \dots, u, \dots, t', \dots, s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots, u, \dots, u', \dots, s_n)}_{e'' \equiv e} \end{aligned}$$

and with  $e$  replaced by  $e'$  the latter becomes:

$$t' = u' \rightarrow \underbrace{\varepsilon_x A(x; s_1, \dots, u, \dots, t', \dots, s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots, t, \dots, u', \dots, s_n)}_{e'}$$

This new formula is provable from  $t = u$  together with

$$\begin{aligned} u = t &\rightarrow \underbrace{\varepsilon_x A(x; s_1, \dots, u, \dots, t', \dots, s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots, t, \dots, t', \dots, s_n)}_{e'''} \\ t' = u' &\rightarrow \underbrace{\varepsilon_x A(x; s_1, \dots, t, \dots, t', \dots, s_n)}_{e'''} = \underbrace{\varepsilon_x A(x; s_1, \dots, t, \dots, u', \dots, s_n)}_{e'} \end{aligned}$$

and some instances of  $(='_2)$ . Hence,  $\pi'$  contains a (possibly new) special  $\varepsilon$ -term  $e''''$ . However,  $e'''' \prec e$  (Exercise: prove this.)

In the special case where  $e = e''$  and  $e' = e'''$ , i.e., the instance of  $(=_\varepsilon)$  we started with, then replacing  $e$  by  $e'$  results in  $t = u \rightarrow e' = e'$ , which is provable from  $e' = e'$ , an instance of  $(=_1)$ .

Let  $\pi_1$  be  $\pi_0$  with the necessary new instances of  $(=_\varepsilon)$ , added. The instances of  $(=_\varepsilon)$  in  $\pi_1$  satisfy the properties required in the statement of the lemma.

However, the results of replacing  $e$  by  $e'$  may have impacted some of the critical formulas in the original proof. For a critical formula to which  $e \equiv \varepsilon_x A(x, u)$  belongs is of the form

$$A(t', u) \rightarrow A(\varepsilon_x A(x, u), u) \quad (1)$$

which after replacing  $e$  by  $e'$  becomes

$$A(t'', u) \rightarrow A(\varepsilon_x A(x, t), u) \quad (2)$$

which is no longer a critical formula. This formula, however, can be derived from  $t = u$  together with

$$\begin{aligned} A(t'', u) &\rightarrow A(\varepsilon_x A(x, t), u) & (\varepsilon) \\ t = u &\rightarrow (A(\varepsilon_x A(x, t), t) \rightarrow A(\varepsilon_x A(x, t), u)) & (=2) \\ u = t &\rightarrow (A(t'', u) \rightarrow A(t'', t)) & (=2) \end{aligned}$$

Let  $\pi_2$  be  $\pi_1$  plus these derivations of (2) with the instances of  $(=_2)$  themselves proved from  $(='_2)$  and  $(=_\varepsilon)$ . The rank of the new critical formulas is the same, so the rank of  $\pi_2$  is the same as that of  $\pi$ . The new instances of  $(=_\varepsilon)$  required for the derivation of the last two formulas only contain  $\varepsilon$ -terms of lower rank than that of  $e$ . (Exercise: verify this.)

$\pi_2$  is thus a proof of  $E$  from  $t = u$  which satisfies the conditions of the lemma. From it, we obtain a proof  $\pi_2[t = u]$  of  $t = u \rightarrow E$  by the deduction theorem. On the other hand, the instance  $t = u \rightarrow e' = e$  under consideration can also be proved trivially from  $t \neq u$ . The proof  $\pi[t \neq u]$  thus is also a proof, this time of  $t \neq u \rightarrow E$ , which satisfies the conditions of the lemma. We obtain  $\pi'$  by combining the two proofs.  $\square$

*Proof.* Proof of the Theorem By repeated application of the Lemma, every instance of  $(=_\varepsilon)$  involving  $\varepsilon$ -terms of a given type  $p$  can be eliminated from the proof. The Theorem follows by induction on the number of different types of special  $\varepsilon$ -terms of rank  $\text{rk}(\pi)$  in  $\pi$ .  $\square$

**Exercise 14.** Prove Proposition 3.6.

**Exercise 15.** Verify that  $\prec$  is a strict total order.

**Exercise 16.** Complete the proof of the Lemma.